

Separating Vector Bundle Sections by Invariant Means

Gestur Ólafsson and Joseph A. Wolf

ABSTRACT. We sharpen the construction of representation space in the paper “Principal Series Representations of Infinite Dimensional Lie Groups II: Construction of Induced Representations”. We show that the principal series representation spaces constructed there, are completions of spaces of sections of Hilbert bundles rather than completions of quotient spaces of sections.

This note is a continuation of [5], using the same notation. We sharpen the construction of the representation spaces in [5, §§5B, 5C] by proving that the bounded right uniformly continuous sections of a homogeneous Hilbert space bundle $\mathbb{E}_\tau \rightarrow G/H$ (defined by a unitary representation τ of H) are separated by means on G/H .

General Setting

G is a topological group, not necessarily locally compact, and H is a closed amenable subgroup. τ is a unitary representation of H , say on E_τ , and $\mathbb{E}_\tau \rightarrow G/H$ is the associated homogeneous Hilbert space bundle. The space $RUC_b(G/H; \mathbb{E}_\tau)$ of bounded right uniformly continuous bounded sections of $\mathbb{E}_\tau \rightarrow G/H$ consists of the right uniformly continuous bounded functions $f : G \rightarrow E_\tau$ such that $f(xh) = \tau(h)^{-1}f(x)$ for $x \in G$ and $h \in H$. G acts on it by $(\pi_\tau(x)f)(x') = f(x^{-1}x')$. Since τ is unitary the pointwise norm $\|f(xH)\|$ is defined. If μ is a mean on G/H we then have a seminorm on $RUC_b(G/H; \mathbb{E}_\tau)$ defined by $\nu_\mu(f) = \mu(\|f\|)$. We denote the space of all means on G/H by $\mathcal{M} = \mathcal{M}(G/H)$.

We use properties of means and amenability from [2], [3] and [4].

PROPOSITION 1. *If $0 \neq f \in RUC_b(G/H; \mathbb{E}_\tau)$ then there exists $\mu \in \mathcal{M} = \mathcal{M}(G/H)$ such that $\nu_\mu(f) \neq 0$. In other words, in [5, Prop. 5.13 and Cor. 5.14], $\Gamma_{\mathcal{M}}(G/H; \mathbb{E}_\tau)$ is the locally convex TVS completion of $RUC_b(G/H; \mathbb{E}_\tau)$.*

PROOF. Let $f \in RUC_b(G/H; \mathbb{E}_\tau)$ be annihilated by all the seminorms ν_μ , $\mu \in \mathcal{M}$. Suppose that f is not identically zero and choose $x \in G/H$ with $f(x) \neq 0$. WE can scale and assume $\|f(x)\| = 1$. Evaluation $\delta_x(\varphi) = \varphi(x)$ is a mean on G and $\delta_x(\|f\|) = 1$. Now the compact convex set $S = \{\sigma \in \mathcal{M}(G) \mid \sigma(\|f\|) = 1\}$

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(weak* topology) is nonempty. Since H is amenable it has a fixed point μ_f on S . Now μ_f is a mean on G/H and the seminorm $\nu_{\mu_f}(f) = 1$. \square

Principal Series

We specialize Proposition 1 to our setting where G is a real Lie group, e.g. $Sp(\infty; \mathbb{R})$, and P is a minimal self-normalizing parabolic subgroup. Then the amenably induced representations $\text{Ind}_P^G(\tau)$ of G on the $\Gamma_{\mathcal{M}}(G/P; \mathbb{E}_{\tau})$, in other words the general principal series representations of G , do not require passage to quotient spaces of the $RUC_b(G/P; \mathbb{E}_{\tau})$. Further, the argument of [5, Proposition 5.16], that $\text{Ind}_P^G(\tau)|_K = \text{Ind}_M^K$ when the parabolic P is flag-closed, is simplified because we need not compare quotient structures.

Other Completions

Here is a Fréchet space completion of $RUC_b(G/P; \mathbb{E}_{\tau})$. Note that $G = \varinjlim G_n$ where the G_n are real reductive groups defined over the rational number field \mathbb{Q} in a consistent way. So we have the rational group $G_{\mathbb{Q}} := \varinjlim G_{n, \mathbb{Q}}$. The point is that the $G_{n, \mathbb{Q}}$ are countable, so $G_{\mathbb{Q}}$ is countable, and the evaluations form a countable family $\{\delta_{xP} \mid x \in G_{\mathbb{Q}}\}$ of means on G/P . If $f \in RUC_b(G/P; \mathbb{E}_{\tau})$ and $\|f\|$ is annihilated by each of the “rational” seminorms $\nu_{\delta_{xP}}$, the argument of Proposition 1 shows that $f = 0$. The locally convex TVS structure of $RUC_b(G/P; \mathbb{E}_{\tau})$, using only that countable family of seminorms, defines a Fréchet space completion of $RUC_b(G/P; \mathbb{E}_{\tau})$. The action of $G_{\mathbb{Q}}$ extends by continuity to this completion of $RUC_b(G/P; \mathbb{E}_{\tau})$, but it is not clear whether the action of G extends.

We enumerate $G_{\mathbb{Q}}$ by the positive integers to define a mean $\mu = \sum_{m \geq 0} 2^{-m} \delta_{x_P}^m$ on G . The corresponding seminorm $\nu_{\mu}(f) = \sum_{m \geq 1} 2^{-m} \|f(x_m)\|$ is a norm on $RUC_b(G/P; \mathbb{E}_{\tau})$. It defines a pre Hilbert space structure on $RUC_b(G/P; \mathbb{E}_{\tau})$ by $\langle f, h \rangle = \sum_{m \geq 1} 2^{-m} \langle f(x_m), h(x_m) \rangle$. Again, the action of G on $RUC_b(G/P; \mathbb{E}_{\tau})$ does not appear to extend by continuity to the corresponding Hilbert space completion.

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DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70803
E-mail address: olafsson@math.lsu.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720–3840
E-mail address: jawolf@math.berkeley.edu